EHD FLOWS AT LARGE ELECTRIC REYNOLDS NUMBERS

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The motion of a medium consisting of neutral particles and charged particles of single sign is studied under the assumption that the electric Reynolds number ($R_q = u/bE$) is large. We calculate the "freezing-in" integral and the Bernoulli and Cauchy-Lagrange integrals, study the fluid motion in a stream tube, and formulate the boundary layer problem.

1. Basic equations and the "freezing-in" integral. For large electric Reynolds numbers (small values of the mobility b) the EHD equations presented in [1] simplify considerably. When b = 0 the equation of motion of the charged component (Ohm's law) has the simple form

$$\mathbf{j} = q\mathbf{u}.$$

It follows from the equations

$$\partial q / \partial t + \operatorname{div} \mathbf{j} = 0, \ \partial \rho / \partial t + \operatorname{div} \rho \mathbf{u} = 0$$

that

$$q = \beta\rho. \tag{1.1}$$

In (1.1) the quantity β is constant in a particle ($d\beta/dt = 0$). In other words, freezing of the charged particles into the neutral medium takes place; in the stationary case β does not change along a streamline.

It is not difficult to see that the electric field intensity vector flux through any closed surface consisting of fluid particles remains constant. In fact, we have

$$\frac{d}{dt} \int\limits_S E_n dS = 4\pi \int\limits_V \left(\frac{\partial q}{\partial t} + \operatorname{div} q\mathbf{u} \right) dV = 0.$$

Introducing the electric field potential, we can write the EHD equations in the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0, \quad \rho \frac{d\mathbf{u}}{dt} = -\nabla p - \beta\rho\nabla\varphi + \operatorname{div} \boldsymbol{\pi}, \tag{1.2}$$

$$\rho c_p \frac{dT}{dt} = \frac{dp}{dt} + \Phi + \operatorname{div} (\lambda \nabla T), \tag{1.3}$$

$$p = \rho RT, \ \triangle\varphi = -4\pi\beta\rho, \ \mathbf{E} = -\nabla\varphi. \tag{1.4}$$

Here π is the viscous stress tensor, Φ is the dissipative function [2].

The system of equations (1.2)–(1.4) is closed. The charge q and the electric field **E** are found after solving (1.1) and the last equation of (1.4). Equations (1.2)–(1.4) are analogous to the equations describing the motion of a medium in a self-consistent gravity field [3].

2. Bernoulli and Cauchy-Lagrange integrals. Let us examine the motion of an inviscid and non-heat-conducting medium. In this case the system of equations (1.2)–(1.4) admits the Bernoulli and Cauchy-Lagrange integrals. Writing the equations of motion in the Gromeka-Lamb form, we have

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla u^2 + \frac{1}{\rho}\nabla p + \operatorname{rot} \mathbf{u} \times \mathbf{u} + \beta\nabla\varphi = 0 \tag{2.1}$$

Let the flow be stationary. Projecting (2.1) onto the arbitrary line L and introducing along this line its the direction of measurement of its length l we obtain

$$\frac{\partial}{\partial l} \left(\frac{u^2}{2} + \mathscr{P} + \beta\varphi \right) + (\operatorname{rot} \mathbf{u} \times \mathbf{u})_l - \varphi \frac{\partial \beta}{\partial l} = 0, \quad \mathscr{P} = \int \frac{dp}{\rho(p, L)} \tag{2.2}$$

If L coincides with a streamline the Bernoulli integral holds

$$u^2/2 + \mathscr{P} + \beta(L)\varphi = C(L). \tag{2.3}$$

In this case C(L), generally speaking, depends on the streamline. We see from (2.2) that if $\varphi\nabla\beta = \operatorname{rot}\mathbf{u}\times\mathbf{u}$, then the constant C in the Bernoulli integral is the same throughout the flow. For β independent of the streamline, the conditions for constance of C throughout the flow coincide with the conditions in conventional hydrodynamics.

Let us assume that the motion is potential $\mathbf{u} = \nabla\psi$ and barotropic $p = p(\rho)$. In this case the Gromeka-Lamb equations are written in the form

$$\nabla\left(\frac{\partial\psi}{\partial t} + \frac{u^2}{2} + \mathscr{P}\right) = -\beta\nabla\varphi. \tag{2.4}$$

The mass force $-\beta\nabla\varphi$ has the potential $U(\nabla U = -\beta\nabla\varphi)$ when $\operatorname{rot}(\beta\nabla\varphi) = \nabla\beta\times\nabla\varphi = 0$. In this case the Cauchy-Lagrange integral holds

$$\frac{\partial\psi}{\partial t} + \frac{u^2}{2} + \mathscr{P} - U = C(t). \tag{2.5}$$

When β is constant throughout the flow, $U = -\beta\varphi$.

Let us examine the motion of an incompressible fluid ($\rho = \rho_0$ = const) with β = const ($q = \beta\rho_0 = q_0$ = const). In this case the problems of determining the motion of the medium and the electric potential separate. Equations (1.2)-(1.4) for steady potential flow are written in the form

$$\Delta\psi = 0,\ \ \Delta\varphi = -4\pi q_0,\ \ 0.5\,u^2 + p/\rho_0 + \beta\varphi = C. \tag{2.6}$$

Let us examine the problem of flow past a body. The problem solution reduces to independent solution of the first two equations of (2.6) with their corresponding boundary conditions. The third equation of (2.6) must be used to calculate the forces acting on the surface of the body. Let us calculate, for example, the force acting on a body as it moves in a given constant electric field E_0 (the electric field can be considered given for small values of the parameter $Q = 4\pi q_0 L/E_0$). By selection of the coordinate system we can always arrange it so that the electric field has only a single component along the z-axis. From the last equation of (1.4) for constant E_0 we have $\varphi = -E_0 z$. We note that for small Q the second equation of (2.6) takes the form $\Delta\varphi = 0$. We see from the third equation of (2.6) that this case is equivalent to the case of body motion in a fluid in a constant gravitational field.

It is easy to show with the aid of the third equation of (2.6) that an additional force equal to $q_0 V E_0$ will act on the body. Here V is the volume of the body. The quantity $q_0 V$ is the total charge of the medium in the volume occupied by the body. Thus an additional force acts on the body which is equal to the product of the electric field by the magnitude of the total charge of the medium in the volume occupied by the body. The direction of this force is opposite the direction of the electric field intensity vector. This conclusion is valid for $q_0 > 0$. If $q_0 < 0$ the direction of action of the force coincides with the direction of the electric field intensity vector.

It is not difficult to see that the derivation of the formula for the Archimedes force remains the same for finite values of the electric Reynolds numbers.

3. Motion in a streamtube. Let us examine the motion of an ideal, non-heat-conducting, perfect gas in a slender tube of variable section. We assume that the flow in such a tube is one-dimensional, i.e., the fluid velocities are approximately the same at different points of the section σ and for steady flow differ only with transition from one section to another. We direct the x-axis along the tube axis. For adiabatic flows the system of equations (1.2)–(1.4) has the following integrals

$$\rho u\sigma = m,\ q u\sigma = I,\ p = C\rho^\gamma,\ m\,(c_p T + 0.5\,u^2) + I\varphi = m i_0 \tag{3.1}$$

Here $\sigma = \sigma(x)$ is the streamtube section area at the point x, m and I are the mass flow rate and charge flow rate (total current) through the tube section, i_0 is the total enthalpy per unit mass. The third equation of (3.1) is the adiabaticity integral, and the fourth is the energy integral. In the following we assume that the streamtube section is known as a function of x, and all the hydrodynamic and electric quantities q_0 and φ_0 are known at the initial section. Relations (3.1) and the first equation of (1.4) make it possible to connect the values of all the parameters at the initial and final sections of the tube if we assume that the value φ of the electric potential at the final section is known and is determined from Ohm's law in the external circuit. To determine the variation of the hydrodynamic and electric quantities along the x-axis we must use, in addition to (3.1) and the first equation of (1.4), the differential equation

$$\frac{m}{\sigma(x)} u' + p' = -\frac{I\rho}{m} \varphi'. \tag{3.2}$$

It is convenient to pose the boundary conditions for the potential φ in the form

$$\varphi = 0 \quad \text{for } x = 0 \quad \varphi = \varphi_1 \quad \text{for } x = L. \tag{3.3}$$

The quantity L is the tube length and φ_1 is found from Ohm's law for the external circuit.

Let us examine fluid flow in a channel of constant section. In this case (3.2) admits the integral

$$mu / \sigma + p - (\varphi')^2 / 8\pi = \Pi = \text{const}. \tag{3.4}$$

We reduce (3.1), (3.4) to dimensionless form

$$\begin{gathered} \rho^* u^* = 1, \quad q^* u^* = 1, \quad p^* = \rho^{*\gamma}, \\ \frac{T^*}{(\gamma - 1) M^2_0} + \frac{u^{*2}}{2} + S\varphi^* = \frac{1}{2} + \frac{1}{(\gamma - 1) M_0^2}, \\ u^* + \frac{p^*}{\gamma M_0^2} = \frac{S}{2Q} [(\varphi^{*\prime})^2 - (\varphi_0{}^{*\prime})^2] + 1 + \frac{1}{\gamma M_0^2}. \end{gathered} \tag{3.5}$$

Sometimes it is also convenient to use the equation $\varphi^{*\prime\prime} = -Q/u^*$.

Then the boundary conditions (3.3) take the form

$$\varphi^* = 0 \quad \text{for } x^* = 0, \quad \varphi^* = 1 \quad \text{for } x^* = 1. \tag{3.6}$$

Here

$$x^* = x / L, \; u^* = u / u_0, \; p^* = p / p_0, \; T^* = T / T_0, \; q^* = q/q_0,$$

$$M_0{}^2 = \frac{u_0{}^2}{\gamma R T_0}, \quad \varphi^* = \frac{\varphi}{\varphi_1}, \quad Q = \frac{4\pi q_0 L^2}{\varphi_1}, \quad S = \frac{q_0 \varphi_1}{\rho_0 u_0{}^2}, \quad E^* = \frac{EL}{\varphi_1}.$$

The magnitude $-(\partial\varphi^*/\partial x^*)_0$ of the electric field at the initial section of the channel which appears in (3.5) is not known and must be found from the problem solution. All the other quantities with zero subscripts are assumed to be known values of the parameters at the initial section.

We present as an example the results of the numerical solution of the system of equations (3.5) with the boundary conditions (3.6). In the calculations we used the following numerical values of the parameters: $S = 0.1$, $Q = 100$, $M_0 = 0.5$, $\gamma = 1.4$. The calculation was made for the case in which $\varphi_1 > 0$. Figures 1 and 2 show the curves of $\chi(\chi = E^*/80)$ and the potential φ^* as a function of x^*. For the selected values of the parameters the electric field vanishes at some section close to the midpoint of the channel and thereafter becomes positive. At this section the generator flow regime changes to the accelerator regime.

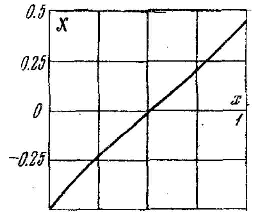

Fig. 1

4. One-dimensional flow in constant-section channel with small interaction parameter. In the case of a small interaction parameter $S \ll 1$ the electric forces have no effect on the hydrodynamic flow. In this case the system of equations (3.5) is satisfied by the solution with constant values of the hydrodynamic parameters u^*, p^*, ρ^*, and T^*. The differential equation for the electric potential takes the form

$$\varphi^{*''} = -Q. \tag{4.1}$$

The solution of (4.1) is

$$\varphi^* = -E_0^* x^* - 0.5Qx^{*2}. \tag{4.2}$$

From the boundary condition (3.6) for $x^* = 1$ we have

$$E_0^* = -(1 + 0.5Q).$$

The expression for the electric field has the form

$$E^* = Qx^* - (1 + 0.5Q). \tag{4.3}$$

Let us study solution (4.2). The value of x^* for which the potential φ^* reaches its maximum value is found from the condition $\varphi^{*\prime} = 0$. We have

$$x_m^* = (1 + 0.5Q)/Q \tag{4.4}$$

At the section $x^* = x_m^*$ the electric field E^* vanishes and there is a change of the flow conditions. Depending on the numerical value of the parameter Q the section $x^* = x_m^*$ may be located at different distances from the channel entrance. It follows from (4.4) that for $Q = 2$ the value of x_m^* equals unity, i.e., this section coincides with the end of the channel. For $Q < 2$ the electric field E^* does not change sign inside the channel. For large values of the parameter Q we have approximately $x_m^* = 0.5$. The calculations presented in section 3 for arbitrary values of the interaction parameter show that this conclusion is confirmed even for $S = 0.1$ and $Q = 100$. Noting that $Q = 4\pi q_0 L^2/\varphi_1$ we obtain the condition for constancy of the sign of E^*, using Ohm's law for the external circuit

$$\varphi_1 = hj_0 r = Rq_0 u_0$$

Here h is the channel height, j_0 is the electric current density, r is the external circuit resistance, and $R = hr$.

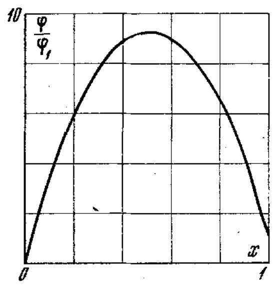

Fig. 2

The condition $Q < 2$ yields

$$u_0 > 2\pi L^2/R.$$

To determine that part of the gas energy which is converted into electric energy, we write the energy integral (3.4) in the form

$$i_0 - i_1 = \varphi_1 j_0/m = \varphi_1^2/mR.$$

Here $i_0 - i_1$ is the difference of the enthalpies at the initial and final channel sections. It is obvious that the maximum value of φ_1 which can be obtained at the exit from the channel is limited by the condition

$$\varphi_1 \leqslant \sqrt{i_0 Rm}.$$

A given power φ_1^2/mR in a channel of given length with given i_0 and j_0 can be obtained in two ways with high flow velocities and low gas temperatures ($i_0 = c_p T + u^2/2$ is given) and with low flow velocities but high gas temperatures at the entrance section.

One-dimensional flows in channels of variable section were examined in [4] in the approximate formulation. In [4], $E = 0$ for $x = L$ was used as the boundary condition. This condition defines the particular solution in which the potential reaches maximum value at the end of the channel.

5. Formulation of the boundary layer problem. The system of equations (1.2)–(1.4) describes the flow of a viscous compressible electrically conducting fluid in an electric field. For large values of the Reynolds number the viscosity and thermal conductivity of the medium need be considered not in the entire flow region, but only in narrow layers in which there is marked variation of the velocity and temperature—the so-called boundary layers. Generally speaking, in EHD there may exist in addition to the viscous and thermal boundary layers a layer of marked variation of the charged particle density, whose thickness may differ from that of the viscous and thermal boundary layers. However, it will be shown later than in a wide range of variation of the defining parameters the thickness of the electric charge density variation layer coincides with the thickness of the ordinary boundary layer.

Let us estimate the boundary layer for the system of equations (1.2)–(1.4). For simplicity we examine the plane case, in which all quantities depend only on the x- and y-coordinates. We introduce the thickness δ of the layer across which there is a marked change of the hydrodynamic quantities. Equating the orders of magnitude of the viscous term and the electric terms in the equation of motion (1.2) in the projection onto the x-axis, we obtain the following estimate:

$$\delta / L \sim (RS)^{-0.5}.$$

Here we have dropped the term $\partial/\partial y\ (\eta \partial u/\partial y)$ in the expression for the viscous stress tensor, as is done in conventional hydrodynamics. It follows from the continuity equation (1.2) that $\rho^* v^* \sim \delta/L$. We write out the equation of motion in the projection onto the x-axis, taking into account the estimates made above

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + qE_x + \frac{\partial}{\partial y}\left(\eta \frac{\partial u}{\partial y}\right). \tag{5.1}$$

For the estimate of the term qE_x in (5.1) we use the "freezing-in" integral $q = \beta\rho$ and the equation

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = 0 \quad (\text{rot}\, \mathrm{E} = 0)\,. \tag{5.2}$$

It follows from (5.2) that ΔE_x (change of E_x across the boundary layer) has the order

$$\triangle E_x \sim \frac{\partial E_y}{\partial x}\delta\,. \tag{5.3}$$

We shall assume that $|\partial E y/\partial x| \lesssim E_0/L$. Then $\Delta E_x \sim \delta$ and the change of E_x across the viscous boundary layer can be neglected, taking

$$E_x = E_x^\infty\,(x)\,. \tag{5.4}$$

Hereafter the infinity symbol will be used to denote values of quantities at the outer edge of the boundary layer. We shall show that the pressure p does not change across the boundary layer. To do this, we use the equation of motion in the projection onto the y-axis, retaining only the principal terms

$$\partial p\,/\,\partial y = q\,E_y \tag{5.5}$$

and the second and third equations of (1.4), which with account for (5.4) can be written in the form

$$\partial E_y\,/\,\partial y = 4\pi\,(q - q_\infty). \tag{5.6}$$

In deriving (5.6) we assumed that $E_y^\infty = 0$. It follows from (5.5) and (5.6) that

$$\frac{\partial}{\partial y}\left(p - \frac{E_y{}^2}{8\pi}\right) = q_\infty E_y\,. \tag{5.7}$$

Integrating (5.7) across the viscous boundary layer, we obtain

$$p - E_y{}^2/8\pi \sim p_\infty. \tag{5.8}$$

We assume that $E_y \lesssim E_0$ is the magnitude of the maximum possible electric field, determined by the magnitude of the breakdown voltage. In this case, we have from (5.8) for not too small pressures in the outer flow that

$$\frac{p}{p_\infty} - 1 \sim \frac{E_y^2}{8\pi p_\infty} \leqslant \frac{E_0^2}{8\pi p_\infty} \ll 1.$$

Thus the pressure variation across the boundary layer can be neglected, taking $p = p_\infty(x)$.

For β = const the q variation thickness coincides with the ρ variation thickness. It follows from the heat-influx equation (1.3), which coincides exactly with the corresponding equation in conventional hydrodynamics, and the equation of state that the charge density q variation thickness coincides with the thermal layer thickness.

We write out the final system of boundary layer equations in the case b = 0

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \quad p = \rho RT,$$
$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -p' + \beta\rho E_x^\infty + \frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}\right),$$
$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = up' + \frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial y}\right)^2. \tag{5.9}$$

The boundary conditions for (5.9) coincide with the boundary conditions used in conventional hydrodynamics. The quantities p and E_x^∞ in (5.9) are determined from the solution of the one-dimensional equations in the outer flow, which were examined in sections 3 and 4 for flow in channels. The study of one-dimensional flows for $b \neq 0$ was presented in [5].

Thus, for the case of large electric Reynolds numbers (small values of the charged particle mobility) the system of boundary layer equations differs from that of conventional hydrodynamics by the presence of an additional term $\beta\rho E_x$ in the momentum equation, in which E_x is found from the solution of the problem in the outer flow.*

In the case ρ = const and β = const (we note that in this case q = const) we can introduce the function ρ^* by the relation

$$p^* = p + q\varphi_\infty(x). \tag{5.10}$$

Here $\varphi_\infty(x)$ is a function which is determined from the solution of the problem in the core flow. In this case (5.9) takes the same form as the corresponding system of boundary layer equations in conventional incompressible fluid hydrodynamics, in which p^* appears in place of p. The equations for the outer flow in the one-dimensional formulation also coincide with the corresponding equations of incompressible fluid hydrodynamics when p is replaced by p^*. The results obtained in conventional hydrodynamics can be used to solve the boundary layer problems in EHD with b = 0, β = const, ρ = const. Here it is not difficult to see that the electric field affects only the distribution of the pressure p in the fluid. The conventional hydrodynamics condition for separation of the boundary layer from the wall [6] can be written in the form

$$\frac{p^{*\prime}\delta^2(x)}{\eta u_\infty} = \frac{(p + q\varphi_\infty)'\delta^2(x)}{\eta u_\infty} = 2. \tag{5.11}$$

An analogous condition for the separation point can also be obtained for the case of compressible gas flow at a thermally isolated wall ($(\partial T/\partial y)_w = 0$). Hereafter we use the symbol w to denote values of all the parameters at the wall. To derive the boundary layer separation condition we shall use the technique presented in [6, 7]. The velocity profile at the separation point is represented in the form of a Taylor series in which the values of the derivatives $\partial^2 u/\partial y^2$, $\partial^3 u/\partial y^3$, ... are found by successive differentiation of the original system of equations and use of the relations ($u = 0$, $v = 0$, $\partial u/\partial y = 0$) at the separation point. Using system (5.9), we can easily show that at the separation point all the terms of the Taylor series equal zero except for the term containing the second derivative. Using the relation

$$\eta_w(\partial^2 u/\partial y^2)_w = p' - \beta\rho_w E_x^\infty$$

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*The question of control of the EHD boundary layer was examined in studies of Kas'yanov and Boyarskii. See V. A. Kas'yanov, Candidate's dissertation, Institute of Hydromechanics, Kiev, 1965; G. N. Boyarskii, Candidate's dissertation, Institute of Hydromechanics, Kiev, 1968.

for the velocity profile at the separation section for β = const, we have

$$u(y) = \frac{y^2}{2\eta_w}(p' - q_w E_x^{\infty}), \quad q_w = \beta\rho_w. \tag{5.12}$$

Setting y = δ in (5.12), we obtain the boundary layer separation condition

$$\frac{\delta^2(x)(p' - q_w E_x^{\infty})}{u_\infty \eta_w} = 2 \tag{5.13}$$

Formula (5.13) permits finding the section at which the boundary layer separates without solving (5.9) if the value of the boundary layer thickness $\delta(x)$ and the value of the wall temperature T_W are found, for example, using semiempirical theories.

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